Stably free modules over virtually free groups*

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Abstract

Let F_m be the free group on m generators and let G be a finite nilpotent group of non square-free order; we show that for each $m \geq 2$ the integral group ring $\mathbf{Z}[G \times F_m]$ has infinitely many stably free modules of rank 1.

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1 Introduction

We study stably free modules over the integral group ring $\mathbf{Z}[G \times F_m]$, where G is a finite group and F_m is the free group on m generators. A finitely generated module P over a ring Λ is said to be stably free when there exists a natural number n such that $P \oplus \Lambda^n \cong \Lambda^m$ for some m. Provided that Λ satisfies the invariant basis number property (see [3]), we may uniquely define the rank of P to be $m-n$. Every integral group ring has the invariant basis number property. We shall prove:

Theorem 1.1. *Let G be a finite nilpotent group of non square-free order. Then there are infinitely many non-isomorphic stably free modules of rank 1 over $\mathbf{Z}[G \times F_m]$ when $m \geq 2$.*

In contrast Johnson [7] has shown that both $\mathbf{Z}[C_p \times F_m]$ and $\mathbf{Z}[D_{2p} \times F_m]$ admit no non-free stably free modules when p is prime, C_p is the cyclic group of order p and D_{2p} is the dihedral group of order $2p$. Johnson [6] has also shown that $k[G \times F_m]$ admits no non-free stably free modules when k is any field and G is any finite group.

In order to produce examples of non-free stably free modules we will use techniques of Milnor [10] for constructing projective modules over fibre product rings. Denote by $\mathcal{SF}_1(\Lambda)$ the set of isomorphism classes of stably free modules of rank 1 over a ring Λ. Our main theorem will be deduced from the following two special cases:

(I): $\mathcal{SF}_1(\mathbf{Z}[C_{p^2} \times F_m])$ is infinite for every prime p and $m \geq 2$;

(II): $\mathcal{SF}_1(\mathbf{Z}[C_p \times C_p \times F_m])$ is infinite for every prime p and $m \geq 2$.

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2 Constructing stably free modules

In [10], Milnor introduced techniques for analysing the structure of projective modules over a fibre product ring in terms of its factors. These techniques were further developed by Swan in [12] to investigate the structure of stably free modules over various group rings. Consider a commutative square of ring homomorphisms

$$\mathcal{A} = \left\{ \begin{array}{ccc} A & \xrightarrow{\pi_-} & A_- \\ \downarrow{\scriptstyle \pi_+} & & \downarrow{\scriptstyle \psi_-} \\ A_+ & \xrightarrow{\psi_+} & A_0 \end{array} \right. \tag{1}$$

such that: (i) A is the fibre product of A_- and A_+ over A_0 : $A = A_- \times_{A_0} A_+$; (ii) ψ_+ is surjective. Such a square will be called a *Milnor square*.

Any right module M over A determines a triple $(M_+, M_-; \alpha(M))$ where $M_\sigma = M \otimes_{\pi_\sigma} A_\sigma$ for $\sigma = +, -$ and $\alpha(M) : M \otimes_{\psi_+\pi_+} A_0 \to M \otimes_{\psi_-\pi_-} A_0$ is an A_0-module isomorphism. Conversely, any triple $(M_+, M_-; \alpha)$ with M_σ a (right) A_σ module for $\sigma = +, -$ and α an isomorphism $\alpha : M_+ \otimes_{\psi_+} A_0 \to M_- \otimes_{\psi_-} A_0$ determines an A module given explicitly as

$$\langle M_+, M_-, \alpha \rangle = \{(m_+, m_-) \in M_+ \times M_- \mid \alpha(m_+ \otimes 1) = m_- \otimes 1\}.$$

The A-action on $\langle M_+, M_-, \alpha \rangle$ is then given by

$$(m_+, m_-) \cdot a = (m_+ \cdot \pi_+(a), m_- \cdot \pi_-(a)).$$

M is said to be *locally free* if both M_+ and M_- are free; since $A_+ \otimes A_0 \cong A_- \otimes A_0 \cong A_0$ the rank of M_+ and M_- are necessarily the same, and we say that the rank of M is this common rank. A locally free module over A is automatically projective (see [10]); the question of when it is stably free is rather more delicate. Denote the set of isomorphism classes of finitely generated locally free modules of rank n over A by $\mathcal{LF}_n(A)$. The triple $(M_+, M_-; \alpha(M))$ associated to M does not completely determine M; however by ([12], Lemma A4) there exists a bijection

$$\mathcal{LF}_n(A) \leftrightarrow \psi_-(GL_n(A_-))\backslash GL_n(A_0)/\psi_+(GL_n(A_+)) \tag{2}$$

Abbreviate the space of double cosets on the right to $\overline{GL_n}(\mathcal{A})$. For each pair of integers $n, k \geq 1$ define a stabilization map

$$\begin{aligned} s_{n,k} : \overline{GL_n}(\mathcal{A}) &\to \overline{GL_{n+k}}(\mathcal{A}) \\ [\alpha] &\mapsto [\alpha \oplus I_k] \end{aligned}$$

Then since the free module A^n determines the triple $(A_+^n, A_-^n; I_n)$, we have that M is stably free if and only if $\alpha = \alpha(M)$ satisfies $s_{n,k}(\alpha) = [I_{n+k}]$ for some k.

The following proposition allows us to construct the original examples of non-trivial stably free modules claimed in **(I)** and **(II)** of the introduction.

Proposition 2.1. *Let $\mathcal{A}$ be a Milnor square as in (1) and suppose that*

$$\psi_-(A_-^*)\backslash[A_0^*, A_0^*]/\psi_+(A_+^*) \tag{3}$$

is infinite, where A_σ^* denotes the unit subgroup of A_σ and $[A_0^*, A_0^*]$ denotes the commutator subgroup of A_0^*. Then $\mathcal{SF}_1(A)$ is infinite.

Proof. Let $\{a_i\}_{i\in I}$ be an infinite set of coset representatives in $\psi_-(A_-^*)\backslash[A_0^*, A_0^*]/\psi_+(A_+^*)$. Each a_i determines an automorphism $(a_i) \in GL_1(A_0)$ and so for each $i \in I$ we may form the locally free A-module $P_i = \langle A_+, A_-; (a_i)\rangle$. Then clearly by (2) $P_i \cong P_j$ if and only if $i = j$. To see that each P_i is stably free, consider $s_{1,1}([a_i]) = \begin{bmatrix} a_i & 0 \\ 0 & 1 \end{bmatrix}$. Each $a_i \in [A_0^*, A_0^*]$ and so by Whitehead's lemma $s_{1,1}([a_i]) \in E_2(A_0)$, where $E_2(A_0)$ is the subgroup of $GL_2(A_0)$ generated by the elementary matrices $E(i,j;a) = I_2 + a\epsilon(i,j)$ $(a \in A_0)$. Since $\psi_+ : A_+ \to A_0$ is surjective, we have an inclusion $E_2(A_0) = \psi_+ E_2(A_+) \subset \psi_+ GL_2(A_+)$ and therefore $s_{1,1}([a_i]) = [I_2] \in \overline{GL_2}(\mathcal{A})$ and so $P_i \oplus A \cong A^2$.

□

3 Stably free cancellation and generalized Euclidean rings

A ring Λ is said to have stably free cancellation (SFC) when every stably free module over Λ is actually free. All principal ideal domains have SFC, as do all local rings. Much of the following discussion is due to Johnson [7].

For any ring Λ, denote by $\mathrm{rad}(\Lambda)$ the Jacobson radical of Λ. Recall that an ideal $\mathbf{m}$ of Λ is said to be *radical* when $\mathbf{m} \subset \mathrm{rad}(\Lambda)$. It is a clear consequence of ([1], p.90, Prop. 2.12) that:

Proposition 3.1. *Let $\mathbf{m}$ be a two sided radical ideal in Λ. Then*

$$\Lambda/\mathbf{m} \textit{ has SFC} \implies \Lambda \textit{ has SFC}$$

Let D be a (possibly non-commutative) division ring. Dicks and Sontag [4] have shown that $D[F_m]$ has SFC. As a consequence of Morita equivalence, a matrix ring $M_n(\Lambda)$ has SFC if and only if Λ has SFC; applying Wedderburn's theorem now shows that $\Lambda[F_m]$ has SFC for any left semi-simple ring Λ. (Note that a product $\Lambda = \Lambda_1 \times \Lambda_2$ has SFC if and only if both Λ_1 and Λ_2 have SFC.) Now suppose that Λ is a left artinian ring. The canonical mapping $\phi : \Lambda \to \Lambda/\mathrm{rad}(\Lambda)$ induces a surjective ring homomorphism $\phi_* : \Lambda[F_m] \to \Lambda/\mathrm{rad}(\Lambda)[F_m]$ in which $\ker(\phi_*) = \mathrm{rad}(\Lambda)[F_m]$. Since Λ is left artinian, $\mathrm{rad}(\Lambda)$ is nilpotent (see Lam [8], Theorem 4.12) and hence $\mathrm{rad}(\Lambda)[F_m]$ is a radical ideal in $\Lambda[F_m]$. Applying 3.1 with $\mathbf{m} = \mathrm{rad}(\Lambda)[F_m]$ now shows:

Corollary 3.2. *Let Λ be a left artinian ring. Then $\Lambda[F_m]$ has SFC.*

For any ring Λ denote by $E_n(\Lambda)$ the subgroup of $GL_n(\Lambda)$ generated by the elementary matrices $E(i,j;\lambda) = I_n + \lambda\epsilon(i,j) \quad (\lambda \in \Lambda)$. Denote by $D_n(\Lambda)$ the subgroup of $GL_n(\Lambda)$ consisting of all diagonal matrices. Λ is said to be *generalized Euclidean* when, for all $n \geq 2$ the following statement holds: for all $A \in GL_n(\Lambda)$ there exists $E \in E_n(\Lambda)$ and $D \in D_n(\Lambda)$ such that $A = DE$; in other words every invertible matrix over Λ is reducible to a diagonal matrix by means of elementary row and column operations. The notion of a generalized Euclidean ring is often useful when dealing with modules over fibre product rings.

A ring homomorphism $\phi : \Lambda_1 \to \Lambda_2$ induces a mapping $\phi : \mathcal{SF}_1(\Lambda_1) \to \mathcal{SF}_1(\Lambda_2)$ given by $\phi(S) = S \otimes_\phi \Lambda_2$. The following is proven in [7]:

Proposition 3.3. *Let A be a Milnor square. If A_0 has SFC and is generalized Euclidean, then the induced map $\pi_+ \times \pi_- : \mathcal{SF}_1(A) \to \mathcal{SF}_1(A_+) \times \mathcal{SF}_1(A_-)$ is surjective.*

A well known theorem of Cohn [2] states that $k[F_m]$ is generalized Euclidean whenever k is a division ring. If Λ is generalized Euclidean, then so is the matrix ring $M_n(\Lambda)$. Since generalized Euclidean rings are closed under products, we have that $\Lambda[F_m]$ is generalized Euclidean whenever Λ is semi-simple.

Proposition 3.4. *If Λ is a ring such that Λ/I is generalized Euclidean for some nilpotent ideal I, then Λ is also generalized Euclidean.*

Proof. Let $A \in GL_n(\Lambda)$ and consider the matrix $\psi_*(A) \in GL_n(\Lambda/I)$, where $\psi_* : GL_n(\Lambda) \to GL_n(\Lambda/I)$ is induced by the mapping $\psi : \Lambda \to \Lambda/I$. Then by hypothesis we may write $\psi_*(A) = DE$ for some $D \in D_n(\Lambda/I)$ and some $E \in E_n(\Lambda/I)$. Choose $\hat{D} \in D_n(\Lambda)$ and $\hat{E} \in E_n(\Lambda)$ such that $\psi_*(\hat{D}) = D^{-1}$ and $\psi_*(\hat{E}) = E^{-1}$; if $X = A\hat{E}\hat{D}$ then clearly $\psi_*(X) = I_n$.

Since $\psi(X_{nn}) = 1$, we have that X_{nn} is a unit as I is nilpotent. Therefore by means of elementary row and column operations we may reduce X so that $X_{rn} = X_{nr} = 0$ for $r \neq n$. Repeating this operation for each diagonal element of X we can reduce X, and hence A, to a diagonal matrix as required. □

As noted above, $\mathrm{rad}(\Lambda)$ is nilpotent whenever Λ is left artinian. Applying 3.4 with $I = \mathrm{rad}(\Lambda)[F_m]$ gives:

Corollary 3.5. *Let Λ be a left artinian ring. Then $\Lambda[F_m]$ is generalized Euclidean.*

4 Proof of I and II

It is easy to show that, if I and J are ideals of a ring R, then

$$\begin{tikzcd} R/(I \cap J) \arrow[r] \arrow[d] & R/J \arrow[d] \\ R/I \arrow[r] & R/(I + J) \end{tikzcd}$$

is a fibre square. For any positive integer d let $c_d(x)$ denote the d^{th} cyclotomic polynomial. From the factorization $(x^{p^2}-1) = c_{p^2}(x)c_p(x)c_1(x) = c_{p^2}(x)(x^p-1)$ we obtain the Milnor square

$$\begin{tikzcd}
\mathbf{Z}[x]/(x^{p^2}-1) \arrow[r] \arrow[d] & \mathbf{Z}[x]/(c_{p^2}(x)) \arrow[d] \\
\mathbf{Z}[x]/(x^p-1) \arrow[r] & \mathbf{Z}[x]/I
\end{tikzcd}$$

where I is the sum of the ideals (x^p-1) and $(c_{p^2}(x))$. However, since $c_{p^2}(x) = (x^{p(p-1)} + x^{p(p-2)} + \ldots + x^p + 1)$, we have

$$p = c_{p^2}(x) - (x^{p(p-2)} + 2x^{p(p-3)} + \ldots + (p-2)x^p + (p-1))(x^p-1),$$

and hence $I = (p, x^p - 1)$. Therefore we may rewrite the above square as

$$\begin{tikzcd}
\mathbf{Z}[x]/(x^{p^2}-1) \arrow[r] \arrow[d] & \mathbf{Z}[x]/(c_{p^2}(x)) \arrow[d] \\
\mathbf{Z}[x]/(x^p-1) \arrow[r] & \mathbf{F}_p[x]/(x^p-1)
\end{tikzcd}$$

Writing $A[C_n] = A[x]/(x^n-1)$ and applying the functor $-\otimes \mathbf{Z}[F_m]$ we obtain another Milnor square:

$$\mathcal{A} = \left\{ \begin{tikzcd}
\mathbf{Z}[C_{p^2} \times F_m] \arrow[r] \arrow[d] & \mathbf{Z}[\zeta_{p^2}][F_m] \arrow[d, "\psi_-"] \\
\mathbf{Z}[C_p \times F_m] \arrow[r, "\psi_+"] & \mathbf{F}_p[C_p \times F_m]
\end{tikzcd} \right.$$

where ζ_{p^2} is a primitive p^2-th root of unity. Since $\mathbf{Z}[\zeta_{p^2}]$ is an integral domain, $\mathbf{Z}[\zeta_{p^2}][F_m]$ has only trivial units (see [11], p.591 and p.598); that is

$$\mathbf{Z}[\zeta_{p^2}][F_m]^* = (\mathbf{Z}[\zeta_{p^2}])^* \times F_m$$

Proposition 4.1. $\mathbf{Z}[C_p \times F_m]^* = (\mathbf{Z}[C_p])^* \times F_m$

Proof. Consider the following fibre square, which arises from the factorization $(x^p-1) = (x-1)c_p(x)$:

$$\begin{tikzcd}
\mathbf{Z}[C_p \times F_m]^* \arrow[r, "\pi_-"] \arrow[d, "\pi_+"] & \mathbf{Z}[\zeta_p][F_m]^* \arrow[d] \\
\mathbf{Z}[F_m]^* \arrow[r] & \mathbf{F}_p[F_m]^*
\end{tikzcd}$$

Let $u \in \mathbf{Z}[C_p \times F_m]^*$. Then $\pi_+(u) \in \mathbf{Z}[F_m]^*$ which has only trivial units; thus

$$u = aw \ + \sum_{g \in F_m - \{w\}} a_g g$$

where $a(1) = \pm 1$, $w \in F_m$ and $a_g \in \mathbf{Z}[C_p] = \mathbf{Z}[x]/(x^p - 1)$. Each a_g is divisible by $(x-1)$ since $a_g \in \ker(\pi_+)$. Now consider

$$\pi_-(u) = a(\zeta_p)w \; + \sum_{g \in F_m - \{w\}} a_g(\zeta_p)g.$$

We cannot have $a(\zeta_p) = 0$, or else $(1 + x + \ldots + x^{p-1})|a \implies p|a(1)$ which is a contradiction. Therefore, since $\mathbf{Z}[\zeta_p][F_m]$ has only trivial units we must have $a_g(\zeta_p) = 0$ for all $g \in F_m - \{w\}$. Therefore both $(1+x+\ldots+x^{p-1})$ and $(x-1)$ divide each a_g and so each $a_g = 0$. □

Proposition 4.2. *If $m \geq 2$ then*

$$X = \psi_-(\mathbf{Z}[\zeta_{p^2}][F_m]^*)\backslash[\mathbf{F}_p[C_p \times F_m]^*, \mathbf{F}_p[C_p \times F_m]^*]/\psi_+(\mathbf{Z}[C_p \times F_m]^*)$$

is infinite.

Proof. Let x be a generator of C_p and define $y = (1-x) \in \mathbf{F}_p[C_p]$; then $y^p = 0$. Let s and t be two generators of F_m and define

$$\delta_n = (1+yt)s^n(1+yt)^{-1}s^{-n} \in [\mathbf{F}_p[C_p \times F_m]^*, \mathbf{F}_p[C_p \times F_m]^*].$$

We claim that $\{\delta_n \mid n \in \mathbf{N}\}$ is a set of distinct coset representatives in X. Suppose that $[\delta_n] = [\delta_m]$; then there exists $u \in \mathbf{Z}[\zeta_{p^2}][F_m]^*$ and $u' \in \mathbf{Z}[C_p \times F_m]^*$ such that $\delta_n = \psi_-(u)\delta_m\psi_+(u')$. In fact since u and u' are necessarily trivial units

$$\delta_n = \psi_-(a)\psi_+(b)w\delta_m v$$

for some $a \in \mathbf{Z}[\zeta_{p^2}]^*$, $b \in \mathbf{Z}[C_p]^*$ and some $w, v \in F_m$. The units of $\mathbf{F}_p[C_p]$ are of the form $c + dy$ where $c \in \mathbf{F}_p^*$ and $d \in \mathbf{F}_p$, as $\mathbf{F}_p[C_p]$ is a local ring with maximal ideal generated by y. Therefore we have

$$(1+yt)s^n(1+yt)^{-1}s^{-n} = (c+dy)w(1+yt)s^m(1+yt)^{-1}s^{-m}v.$$

Expanding both sides and comparing coefficients of y^0 we have: $d = 0$, $c = 1$ and $w^{-1} = v$. Comparing coefficients of y^1 gives

$$t - s^n t s^{-n} = wtw^{-1} - ws^m t s^{-m} w^{-1},$$

and so we must have

$$t = wtw^{-1} \text{ and } s^n t s^{-n} = ws^m t s^{-m} w^{-1}.$$

The first equation shows that $w = 1 (= v)$ and the second shows that $m = n$; therefore the δ_i form a set of distinct coset representatives. □

Putting (4.2) and (2.1) together proves (**I**) of the introduction:

Proposition 4.3. *For every prime p and every $m \geq 2$, $\mathcal{SF}_1(\mathbf{Z}[C_{p^2} \times F_m])$ is infinite.*

The proof of (**II**) is very similar. Let A be a ring and consider the Milnor square

$$\begin{array}{ccc} A[x]/(x^p-1) & \longrightarrow & A[x]/(1+x+\ldots+x^{p-1}) \\ \downarrow & & \downarrow \\ A & \longrightarrow & A/p \end{array} \tag{4}$$

Setting $A = \mathbf{Z}[y]/(y^p-1)$ we have

$$\begin{array}{ccc} \mathbf{Z}[x,y]/(x^p-1, y^p-1) & \longrightarrow & \mathbf{Z}[x,y]/(\Sigma_x, y^p-1) \\ \downarrow & & \downarrow \\ \mathbf{Z}[y]/(y^p-1) & \longrightarrow & \mathbf{F}_p[y]/(y^p-1) \end{array}$$

where $\Sigma_x = 1+x+\ldots x^{p-1}$. Making the identifications $\mathbf{Z}[x,y]/(x^p-1, y^p-1) = \mathbf{Z}[C_p \times C_p]$, $\mathbf{Z}[x,y]/(\Sigma_x, y^p-1) = \mathbf{Z}[\zeta_p][C_p]$ and $B[y]/(y^p-1) = B[C_p]$ and tensoring with $\mathbf{Z}[F_m]$ we have

$$\mathcal{B} = \left\{ \begin{array}{ccc} \mathbf{Z}[C_p \times C_p \times F_m] & \longrightarrow & \mathbf{Z}[\zeta_p][C_p \times F_m] \\ \downarrow & & \downarrow \phi_- \\ \mathbf{Z}[C_p \times F_m] & \xrightarrow{\phi_+} & \mathbf{F}_p[C_p \times F_m] \end{array} \right.$$

We first need to show that $\mathbf{Z}[\zeta_p][C_p \times F_m]$ has only trivial units:

Proposition 4.4. $\mathbf{Z}[\zeta_p][C_p \times F_m]^* = (\mathbf{Z}[\zeta_p][C_p])^* \times F_m$.

Proof. Consider the Milnor square formed by setting $A = \mathbf{Z}[y]/(1+y+\ldots+y^p) = \mathbf{Z}[y]/(\Sigma_y)$ in (4), tensoring with $\mathbf{Z}[F_m]$ and then taking unit groups:

$$\begin{array}{ccc} \mathbf{Z}[x,y]/(x^p-1, \Sigma_y)[F_m]^* & \longrightarrow & \mathbf{Z}[x,y]/(\Sigma_x, \Sigma_y)[F_m]^* \\ \downarrow & & \downarrow \\ \mathbf{Z}[y]/(\Sigma_y)[F_m]^* & \longrightarrow & \mathbf{F}_p[y]/(\Sigma_y)[F_m]^* \end{array}$$

Since both $\mathbf{Z}[x,y]/(\Sigma_x, \Sigma_y)$ and $\mathbf{Z}[y]/(\Sigma_y)$ are integral domains the corresponding corners have only trivial units. A similar proof to that of (4.1) now applies. $\square$

Essentially the same proof as that of (4.2) proves:

Proposition 4.5. *If $m \geq 2$ then*

$$\phi_-(\mathbf{Z}[\zeta_p][C_p \times F_m]^*) \backslash [\mathbf{F}_p[C_p \times F_m]^*, \mathbf{F}_p[C_p \times F_m]^*] / \phi_+(\mathbf{Z}[C_p \times F_m]^*)$$

is infinite.

Together (4.5) and (2.1) prove (**II**) of the introduction:

Proposition 4.6. *For every prime p and every $m \geq 2$, $\mathcal{SF}_1(\mathbf{Z}[C_p \times C_p \times F_m])$ is infinite.*

5 Proof of main theorem

Let G be a finite group and let H be a normal subgroup of G. We may form the Milnor square:

$$\begin{tikzcd} \mathbf{Z}[G] \arrow[r] \arrow[d] & \mathbf{Z}[G]/(\Sigma_H) \arrow[d] \\ \mathbf{Z}[G/H] \arrow[r] & (\mathbf{Z}/N)[G/H] \end{tikzcd}$$

where $\Sigma_H = \sum_{h \in H} h$ and $N = |H|$. Tensoring with $\mathbf{Z}[F_m]$ we have:

$$\begin{tikzcd} \mathbf{Z}[G \times F_m] \arrow[r] \arrow[d] & \mathbf{Z}[G \times F_m]/(\Sigma_H) \arrow[d] \\ \mathbf{Z}[G/H \times F_m] \arrow[r] & (\mathbf{Z}/N)[G/H \times F_m] \end{tikzcd}$$

Now by (3.2) and (3.5), $(\mathbf{Z}/N)[G/H \times F_m]$ has SFC and is generalized Euclidean. Hence by (3.3) the induced map

$$\mathcal{SF}_1(\mathbf{Z}[G \times F_m]) \to \mathcal{SF}_1(\mathbf{Z}[G/H \times F_m]) \times \mathcal{SF}_1(\mathbf{Z}[G \times F_m]/(\Sigma_H))$$

is surjective and thus:

Proposition 5.1. *Let G be a finite group with normal subgroup $H \lhd G$. If $\mathcal{SF}_1(\mathbf{Z}[G/H \times F_m])$ is infinite then so is $\mathcal{SF}_1(\mathbf{Z}[G \times F_m])$.*

Let G be a finite group of order p^k where p is prime and $k \geq 2$. Then there exists a normal subgroup $H \lhd G$ such that $|H| = p^{k-2}$ (see [5] p.24). Hence either $G/H \cong C_{p^2}$ or $G/H \cong C_p \times C_p$; in either case by (4.3), (4.6) and (5.1) $\mathcal{SF}_1(\mathbf{Z}[G \times F_m])$ is infinite.

Now let G be a finite nilpotent group of non square-free order. Since G is nilpotent, G is the direct product of its Sylow subgroups (see [5], p.24) say $G \cong H_1 \times \ldots \times H_r$. As $|G|$ is non square-free we may choose a prime p such that p^k is the largest power of p dividing $|G|$ and where $k \geq 2$. Therefore at least one of the H_i has order p^k — assume without loss of generality that $|H_1| = p^k$. Then $|G/(H_2 \times \ldots \times H_r)| = p^k$ and so by (5.1) this completes the proof of our main theorem (1.1).

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